7.5 (continued)

Last time: $\mathcal{L}\{u_c(t)\cdot f(t-c)\} = e^{-cs}\underbrace{\mathcal{L}\{f(t)\}}$

transform of $f(t-c)$ shifted
LEFT by c : $t \to t+c$

for example, $\mathcal{L}\{u_{10}(t)\, e^{-2t}\}$

shift LEFT by 10 : $t \to t+10$

$$= e^{-10s}\,\mathcal{L}\{e^{-2(t+10)}\}$$

$$= e^{-10s}\, e^{-20}\,\mathcal{L}\{e^{-2t}\}$$

$$= e^{-10s}\, e^{-20}\,\frac{1}{s+2}$$

t to s : shift LEFT ($t \to t+c$), transform, $u_c \to e^{-cs}$

back to s is everything opposite way

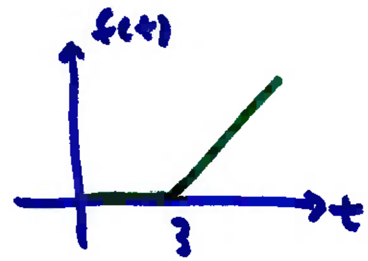
s to t : $e^{-cs} \to u_c$, inverse transform, shift RIGHT ($t \to t-c$)

for example, $\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{2}{s^3}\right\}$ $\qquad$ $\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = t^2$

$= u_\pi(t)\cdot(t-\pi)^2$ $\qquad$ shifted RIGHT by π

let's solve this : $\quad y'' + y = f(t) \qquad y(0) = y'(0) = 0$

$$f(t) = \begin{cases} 0 & 0 \le t < 3 \\ t-3 & t \ge 3 \end{cases}$$



$= u_3(t)\cdot(t-3)$

shift LEFT by 3

$t \to t+3$

transform both sides

$$s^2Y - sy(0) - y'(0) + Y = e^{-3s}\mathcal{L}\{(t+3)-3\}$$

$$s^2Y + Y = e^{-3s}\mathcal{L}\{t\}$$

$$(s^2+1)Y = e^{-3s}\frac{1}{s^2}$$

$$Y = e^{-3s}\frac{1}{s^2(s^2+1)} \longrightarrow \frac{As+B}{s^2} + \frac{Cs+D}{s^2+1} = \dots = \frac{1}{s^2} - \frac{1}{s^2+1}$$

preliminary inv. transform

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\} = t - \sin(t)$$

back to t: $e^{-3s} \to u_3$

inv. transform, then shift RIGHT by 3: $t \to t-3$

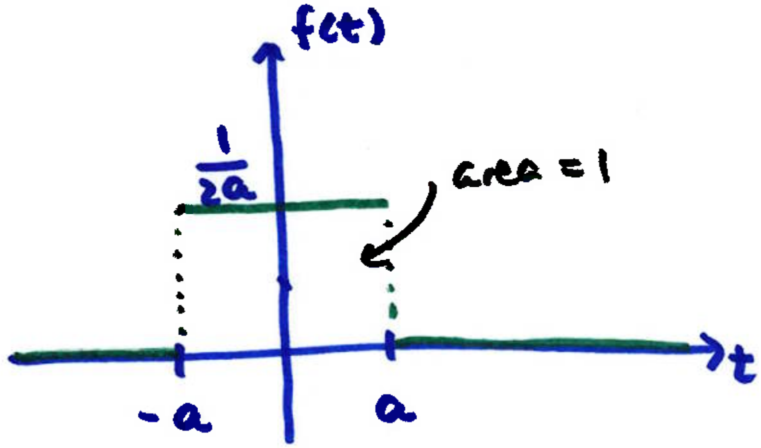
$$y(t) = u_3 \cdot \left[(t-3) - \sin(t-3)\right]$$

$$= \begin{cases} 0 & 0 < t < 3 \\ t-3-\sin(t-3) & t \geq 3 \end{cases}$$

7.6 Impulse Function

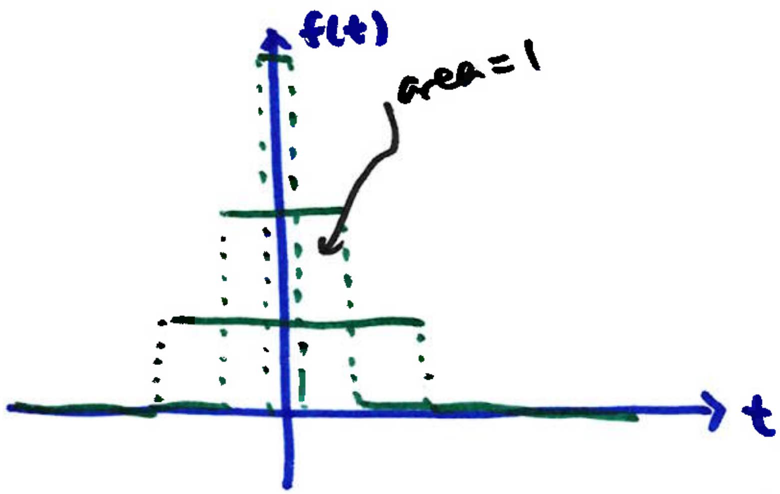
used to model a short-acting input (e.g. hitting a baseball)

we can build this using a step up and then a step down

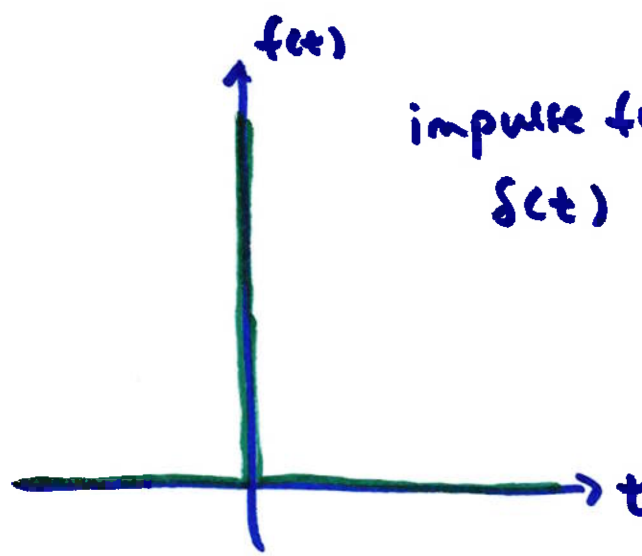


$$f(t) = \frac{1}{2a}\left[u_{-a} - u_a\right]$$

shrink a: $\lim_{a \to 0} f(t)$



$\rightarrow$



impulse function

$\delta(t)$

$$\delta(t) = \begin{cases} +\infty & t = 0 \\ 0 & \text{else} \end{cases} \qquad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$$\delta(t-c) = \begin{cases} +\infty & t = c \\ 0 & \text{else} \end{cases}$$

let's find $\mathcal{L}\{\delta(t-c)\}$

$$= \mathcal{L}\left\{ \lim_{a \to 0} \frac{1}{2a} \left[u_{c-a} - u_{c+a} \right] \right\}$$

$$= \lim_{a \to 0} \frac{1}{2a} \left[\mathcal{L}\{u_{c-a}\} - \mathcal{L}\{u_{c+a}\} \right]$$

$$= \lim_{a \to 0} \frac{1}{2a} \left[\frac{e^{-(c-a)s}}{s} - \frac{e^{-(c+a)s}}{s} \right]$$

$$= \frac{1}{s} \lim_{a \to 0} \frac{1}{2a} \left[e^{-cs} e^{as} - e^{-cs} e^{-as} \right]$$

$$= \frac{e^{-cs}}{s} \lim_{a \to 0} \frac{e^{as} - e^{-as}}{2a}$$ l'Hospital's Rule

$$= \frac{e^{-cs}}{s} \lim_{a \to 0} \frac{s e^{as} + s e^{-as}}{2}$$

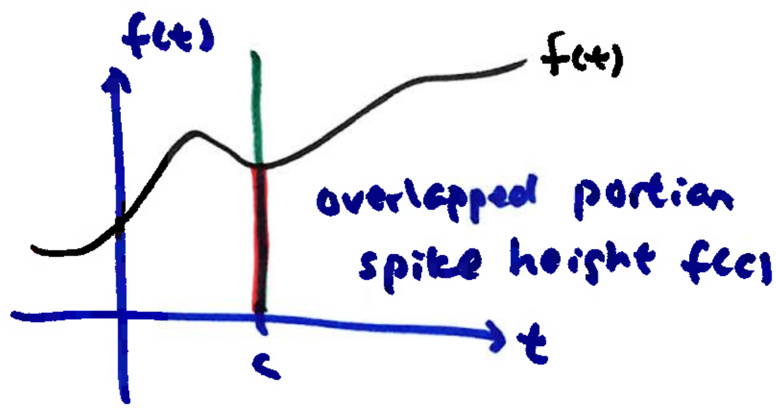
$$= \frac{e^{-cs}}{s} \frac{2s}{2} = e^{-cs}$$

$$\boxed{\mathcal{L}\{\delta(t-c)\} = e^{-cs}}$$

looks similar to $\mathcal{L}\{u_c\} = \frac{e^{-cs}}{s}$

$$\delta(t-c) = \begin{cases} \infty & t = c \\ 0 & \text{else} \end{cases}$$

$$\delta(t-c)\, f(t) = \begin{cases} f(c) & t = c \\ 0 & \text{else} \end{cases}$$



δ "samples" $f(t)$ at $t = c$

$$\mathcal{L}\{\delta(t-c)\, f(t)\} = f(c)\, e^{-cs}$$

revisit earlier mass-spring

$$y'' + y = f(t) \qquad y(0) = y'(0) = 0$$

$$f(t) = \delta(t-3)$$

$$y'' + y = \delta(t-3)$$

$$s^2 Y + Y = e^{-3s}$$

$$Y = e^{-3s} \frac{1}{s^2+1}$$

back to t: $e^{-3s} \to u_3$

$$\mathcal{L}^{-1}\left\{ \frac{1}{s^2+1} \right\} = \sin(t)$$

shift RIGHT by 3

$$t \mapsto t-3$$

$$y(t) = u_3 \cdot \sin(t-3)$$

← not $\delta(t-3)$ in t domain

back to u_3 because the effect is long-lasting even though input is short-acting (baseball continues to fly and doesn't drop to ground right away)